113 Class Problems: Subgroups and Lagrange

1. Let $\mathbb{Z}/45\mathbb{Z}$ be a group under addition. Give an explicit description of a subgroup of size 9. Carefully check that it is indeed a subgroup.

   Solution:

$$H = \{[5a] \mid a \in \mathbb{Z}\} = \{[0],[5],[10],[15],[20],[25],[30],[35],[40]\}$$

- $[0] \in H$
- $[5a],[5b] \in H \Rightarrow [5a]+[5b] = [5(a+b)] \in H$
- $[5a] \in H \Rightarrow [-5a] = [5(-a)] \in H$

2. Let $(G,*)$ be a group. If $H, K \subset G$ are two subgroups prove that $H \cap K$ is a subgroup. If $|H| = 12$ and $|K| = 35$ prove that $H \cap K = \{e\}$. Does there exists $g \in G$ such that $g \notin H \cup K$. Carefully justify your answers.

   Solution:

- $e \in H,\ e \in K \Rightarrow e \in H \cap K$
- $a,b \in H\cap K \Rightarrow a,b \in H$ and $a,b \in K \Rightarrow a*b \in H$ and $a*b \in K$ $\Rightarrow a*b \in H \cap K$
- $a \in H \cap K \Rightarrow a \in H$ and $a \in K \Rightarrow a^{-1} \in H$ and $a^{-1} \in K$ $\Rightarrow a^{-1} \in H \cap K$

subgroup

subgroup

$H \cap K \subset H \Rightarrow |H\cap K| \mid |H|$, $H \cap K \subset K \Rightarrow |H \cap K| \mid |K|$

$HCF(12,35) = 1 \Rightarrow |H \cap K| = 1 \Rightarrow H \cap K = \{e\}$

$\Rightarrow |H \cup K| = 12 + 35 - 1 = 46$

If $G = H \cup K \Rightarrow 12 \mid 46$. Contradiction. Hence $\exists g \in G$ such that $g \notin H \cup K$

3. Give an example of a subgroup of $(\mathbb{R}^2, +)$ which is not a subspace. Justify your answer.

   Solution:

Need a subset which is not closed under scalar multiplication by all of $\mathbb{R}$. For example

$$(\mathbb{Z}^2, +) \subset (\mathbb{R}^2, +)$$

4. Let $(G, *)$ be a group with subgroup $H \subset G$. A **right cosets** of H in G is a subset of the form

$$Hx := \{h * x | h \in H\},$$

   for some $x \in G$. Prove that the right cosets form a partition of G.

   Warning: In general this is a different partition than the left cosets, meaning there exists some $x \in G$ such that $xH \neq Hx$.

   Solution:

- $e \in H \Rightarrow e * x = x \in Hx \Rightarrow \bigcup_{x \in G} Hx = G$
- Let $x, y \in G$ such that $Hx \cap Hy \neq \emptyset$

$\Leftrightarrow \exists h_1, h_2 \in H$ such that $h_1 x = h_2 y$

$\Leftrightarrow \exists h_1, h_2 \in H$ such that $x * y^{-1} = h_1^{-1} * h_2 \in H$

Hence $Hx \cap Hy \neq \emptyset \Leftrightarrow x * y^{-1} \in H$

Let $x * y^{-1} = h \in H \Rightarrow x = h * y$ and $y = h^{-1} * x$

Given $k \in H$ $\quad k * x = (k * h) * y \in Hy \Rightarrow xH \subset yH$

Similarly $\quad k * y = (k * h^{-1}) * x \in Hx \Rightarrow yH \subset xH$

$\Rightarrow xH = yH$